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FINITE HURWITZ BRAID GROUP ACTIONS FOR ARTIN GROUPS

BY

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ABSTRACT

We prove that (with two possible exceptions) the Hurwitz braid group action on the sequence of standard generators of an irreducible Artin group has a finite orbit if and only if the Artin group is of finite type (i.e., the corresponding Coxeter group is finite).

1. Introduction

For $n \geq 2$ and an $n \times n$ integral symmetric matrix $M = (m_{ij})$, where $m_{ii} = 1, m_{ij} \geq 2, 1 \leq i \neq j \leq n$, we define two groups:

$$C(M) = \langle s_1, \ldots, s_n | s_1^2, \ldots, s_n^2, (s_i s_j)^{m_{ij}}, 1 \leq i \neq j \leq n \rangle;$$
$$A(M) = \langle t_1, \ldots, t_n | \pi(t_i, t_j, m_{ij}) = \pi(t_j, t_i, m_{ij}), 1 \leq i \neq j \leq n \rangle.$$

Here $\pi(t_i, t_j, m_{ij})$ is the product $t_i t_j t_i \cdots$ of length m_{ij} and $C(M)$ and $A(M)$ are called the **Coxeter** and **Artin groups** (respectively) **associated to** M. Here $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ are called the **standard generators** for $C(M)$ and $A(M)$. There is an epimorphism

$$\Pi_M : A(M) \to C(M); \quad \Pi_M(t_i) = s_i, 1 \leq i \leq n.$$

If $C(M)$ is a finite group, then we say that $C(M)$ and $A(M)$ have **finite type**. The Coxeter or Artin group will be called **irreducible** if there is no non-trivial partition $\{1, \ldots, n\} = X \cup Y$ where $m_{ij} = 2$ for all $i \in X, j \in Y$. There is a well-known classification [Hu, GB] of finite irreducible Coxeter groups, each such

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corresponding matrix being (associated to a graph) in one of the finite number of families $A_n, B_n, D_n, E_n, F_n, G_n, H_n = I_n$, indicated in Figure 1. Here the vertices of the graphs are labeled $1, \dots, n$ and the matrix associated to such a graph has ij entry 2 if there is no edge between vertices i, j; 3 if there is an unlabeled edge between vertices i, j; k if there is an edge labeled k between vertices i, j. The corresponding matrix will be denoted $M(A_n), M(B_n), \dots$ and the groups by $C(A_n), C(B_n), \dots, A(A_n), A(B_n), \dots$.

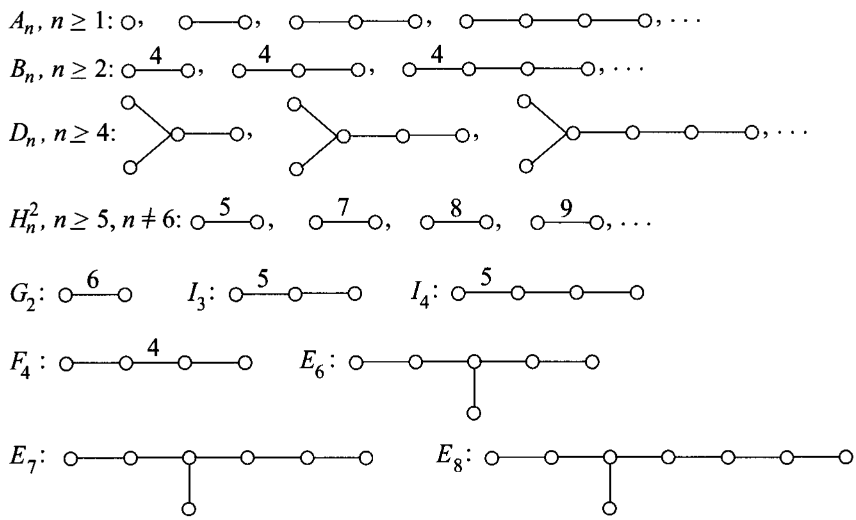

Figure 1.

The group $A(A_{n-1})$ is known as the braid group $\mathcal{B}_n$ on n strands and has many manifestations and applications [Bi]. In this paper we are concerned with the action of $\mathcal{B}_n$ on n-tuples of a given group G. The braid group $\mathcal{B}_n$ has standard generators also denoted by $\sigma_1, \dots, \sigma_{n-1}$. Here σ_i acts on $(g_1, \dots, g_n) \in G^n$ is as follows:

$$\begin{aligned} \sigma_i(\dots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \dots) &= (\dots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \dots), \\ \sigma_i^{-1}(\dots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \dots) &= (\dots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \dots). \end{aligned} \tag{1.1}$$

This is called the **Hurwitz braid group** action and has the property that it preserves the **Coxeter element**: $\Pi_n(g_1, \dots, g_n) = g_1 g_2 \cdots g_n$. For applications of the Hurwitz braid group action see for example [Br, M, CP, CW, Hur, Lo, MT, Co, CMW]. Of course, if the group G is finite, then any element $(g_1, \dots, g_n) \in G^n$ has a finite orbit and so determines a finite permutation representation of

$\mathcal{B}_n$. Even in the case where the orbit is not finite one can sometimes characterise the image; for example, if $G = H_3$ is the integral 3×3 Heisenberg group with generators $a = E_{12}, b = E_{23}, c = E_{13}$, then the action of $\mathcal{B}_3$ on the orbit of (a, b, c) is an extension of S_3 by $\mathbb{Z}^3$ (here E_{ij} is the matrix differing from the identity only in the ij position, which is a 1). We noted in [Hum] that one can sometimes get a finite orbit for a particular sequence of generators of G even if the group G is infinite.

In this paper we look at a special case where G is infinite, namely where G is itself an Artin group.

In [Hum] we considered the Hurwitz action on n-tuples of Euclidean reflections and proved:

(1) Let $n \geq 3, v_1, v_2, , \ldots, v_n \in \mathbb{R}^n$ with $v_1, \ldots, v_n$ linearly independent. Then the $\mathcal{B}_n$-orbit of $(r_{v_1}, r_{v_2}, \ldots, r_{v_n})$ is finite if and only if the group

$$\langle r_{v_1}, r_{v_2}, \ldots, r_{v_n} \rangle$$

is finite if and only if the element $\Pi_n(r_{v_1}, r_{v_2}, \ldots, r_{v_n}) = r_{v_1} r_{v_2} \ldots r_{v_n}$ has finite order. Here r_v is the reflection associated to the vector $v \in \mathbb{R}^n \setminus \{0\}$.

(2) Let $n \geq 1$ and let $W = \langle S \rangle$ be a finite reflection group. If W has type A_n or B_n, then $\mathcal{B}_n$ acts transitively on $\mathcal{S}(W)$, the set of conjugacy classes of n-tuples of reflections generating W. If W has type $D_n, n \geq 4$, then $\mathcal{B}_n(D_n)$ has $n - 2$ orbits for its action on $\mathcal{S}(W)$. These orbits are distinguished by their Coxeter elements (which are not conjugate). The other cases are also determined.

(3) Consider the $\mathcal{B}_n$ action on $\mathcal{S}(\langle S \rangle)$ where $\langle S \rangle = S_{n+1}$. Then the action gives a representation of $\mathcal{B}_n$ of degree $(n+1)^{n-2}$ which is 2-transitive for all $n \geq 3$. If $(n+1)^{n-2}$ is not of the form

$$q, \quad \frac{q^d - 1}{q - 1} \quad (\text{for } d \geq 2), \quad 2^{d-1}(2^d \pm 1) \quad (\text{for } d \geq 3),$$

where q is a prime power, then the $\mathcal{B}_n$ action is that of the symmetric or alternating group of degree $(n+1)^{n-2}$.

For any G, if $(g_1, \ldots, g_n)$ is a sequence of generators for G whose orbit under the Hurwitz braid group action is finite, then we will call the elements that occur as entries in this finite orbit **special Hurwitz generators**. In this paper we consider the Hurwitz action on certain n-tuples of elements of Artin groups. The case $n = 2$ is trivial.

Although the groups $\mathcal{B}_n$ and $A(A_{n-1})$ are isomorphic we will adopt the following convention: we will use $\mathcal{B}_n$ to denote the braid group acting on G^n with the

Hurwitz action as described above, while $A(A_n)$ will denote one of the groups G such that $\mathcal{B}_n$ acts on G^n.

THEOREM 1.1: *Let $n \geq 3$. Let $A(M)$ be an irreducible Artin group with standard generators $t_1, \ldots, t_n$. Then (with the possible exception of types E_7, E_8) the Hurwitz braid group action on the n-tuple $(t_1, \ldots, t_n)$ has a finite orbit if and only if $A(M)$ is of finite type.*

For type A_n we get $(n+1)^{n-1}$ elements in the orbit. There are $\binom{n+1}{2}$ special Hurwitz generators in this case.

For type B_n we get n^n elements in the orbit. There are n^2 special Hurwitz generators in this case.

For type D_n we get $2(n-1)^n$ elements in the orbit. There are $n(n-1)$ special Hurwitz generators in this case.

For type I_3 we get 50 elements in the orbit. There are 15 special Hurwitz generators in this case. The image of $\mathcal{B}_3$ gives a permutation group of order $5^9 \times 10!$; there is a set of blocks of size 5 on which the action is that of S_{10}.

For type I_4 we get 1350 elements in the orbit. There are 60 special Hurwitz generators in this case. The image of $\mathcal{B}_4$ gives a permutation group of order $15^{89} \times 90!$; there is a set of blocks of size 15 on which the action is that of S_{90}.

For type F_4 we get 432 elements in the orbit. There are 24 special Hurwitz generators in this case. The image of $\mathcal{B}_4$ gives a permutation group of order $2^{20}3^{10}$.

For type E_6 we get $41472 = 2^9 3^4$ elements in the orbit. There are 36 special Hurwitz generators in this case.

We note that in all of the above cases of type X_n the number of special Hurwitz generators is equal to the number of positive roots in a root system of type X_n [Hu; p. 80].

In the Coxeter group/reflection group case we showed in [Hum] that certain types X_n had many different sequences of n generating reflections that gave different $\mathcal{B}_n$ orbits. These sequences generally had non-conjugate Coxeter elements. In the Artin group case we have similarly been able to show that there are other sets of sequences of conjugates of the generators $t_1, \ldots, t_n$ that give different finite $\mathcal{B}_n$ orbits. These sequences are not necessarily a set of generators for the Artin group. We list some of these in

THEOREM 1.2: *For type D_5 the orbit of $(a, b, c, d, b^{-1}c^{-1}ecb)$ has $2592 = 2 \times 6^4$ elements. There are 20 special Hurwitz generators in this case. The image of $\mathcal{B}_5$ gives a permutation group of order $12^{215} \times 216!$; there is a set of blocks of size 12 on which the action is that of A_{216}.*

For type I_3 the orbit of $(a, b, c^{-1}b^{-1}a^{-1}cbc^{-1}abc)$ has 54 elements. There are 15 special Hurwitz generators in this case. The image of $\mathcal{B}_3$ gives a permutation group of order $6^8 \times 9!$; there is a set of blocks of size 6 on which the action is that of A_9.

For type I_3 the orbit of $(a, c, c^{-1}b^{-1}a^{-1}cbc^{-1}abc)$ has 50 elements. There are 15 special Hurwitz generators in this case. The image of $\mathcal{B}_3$ gives a permutation group of order $5^9 \times 10!$; there is a set of blocks of size 5 on which the action is that of S_{10}.

These computations were carried out using Magma [MA].

The following gives a connection between Hurwitz actions on finite groups and isomorphic actions on infinite groups.

THEOREM 1.3: *Let G be a group generated by $\underline{g} = (g_1, \ldots, g_n)$. Then there is an infinite group Q with generators $\underline{q} = (q_1, \ldots, q_n)$ and an epimorphism $\pi: Q \to G, \pi(q_i) = g_i, i \leq n$, such that the $\mathcal{B}_n$ action on the orbit $\mathcal{B}_n(\underline{q})$ and the $\mathcal{B}_n$ action on the orbit $\mathcal{B}_n(\underline{g})$ are permutation isomorphic, this isomorphism being induced by π. In particular, $\mathcal{B}_n(\underline{q})$ is finite if G is finite.*

2. The A_n case

In this section we prove Theorem 1.1 for $A(A_n)$. However, we first note that if $A(M)$ is not of finite type, then the corresponding Hurwitz action on the n-tuple $(t_1, \ldots, t_n)$ does not have a finite orbit. This follows from [Hum] since Π_M is a $\mathcal{B}_n$ epimorphism and $\Pi_M(t_1, \ldots, t_n) = (s_1, \ldots, s_n)$ has an infinite orbit [Hum; Theorem 1.1]. We thus see that the Hurwitz action on $(t_1, \ldots, t_n)$ must also have an infinite orbit.

So now assume that M has type A_n. Then $t_i = \sigma_i, 1 \leq i \leq n$. Recall [Bi] that we can faithfully represent the braid group $\mathcal{B}_{n+1} \cong A(A_n)$ as the mapping class group of the $n+1$-punctured disc $\mathcal{D}_{n+1}$ with punctures $\pi_1, \ldots, \pi_{n+1}$, so that σ_i is a half-twist about an arc a_i joining the punctures π_i, π_{i+1}. See Figure 2, where we have indicated the arcs $a_i, 1 \leq i < n$, together with generators x_i of the free fundamental group $\pi_1(\mathcal{D}_n; p) \cong F_n = \langle x_1, \ldots, x_n \rangle$. As the mapping class group, $A(A_n)$ acts on the free group F_{n+1}, the action being given by (1.1) (with $g_i = x_i$).

We will think of $\mathcal{D}_{n+1}$ as a subset of $\mathbb{R}^2$ with the punctures all on the x-axis. Note that the $t_i (= \sigma_i)$ are all conjugate (since $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1} = \sigma_{i+1}$ follows from the braid relations [Bi]) and that every conjugate of every σ_i is a positive half-twist relative to some arc of $\mathcal{D}_{n+1}$ joining two punctures. Thus if

$\alpha \in \mathcal{B}_n$, where $\alpha(t_1, \ldots, t_n) = (y_1, \ldots, y_n)$, represents the Hurwitz action of α on $(t_1, \ldots, t_n) \in A(A_n)^n$, then each y_i can be represented by an arc $a(y_i) = a_i(\alpha)$ joining two punctures on $\mathcal{D}_{n+1}$, where y_i is the positive half twist relative to this arc. Thus $(y_1, \ldots, y_n)$ can be represented as n arcs of $\mathcal{D}_{n+1}$, each arc $a(y_i) = a_i(\alpha)$ being labeled with the label i. The set of such labeled arcs for this $\alpha \in \mathcal{B}_n$ will be denoted $\mathfrak{A}(\alpha)$, which we will also think of as an edge-labeled graph in $\mathcal{D}_{n+1}$ with vertices the punctures π_i. Note that $\mathfrak{A}(\alpha)$ determines α. For example, if $\alpha = id$, then $a_i(id) = a_i$ for $1 \leq i \leq n$ and $\mathfrak{A}(\alpha) = (a_1, \ldots, a_n)$ is shown in Figure 2.

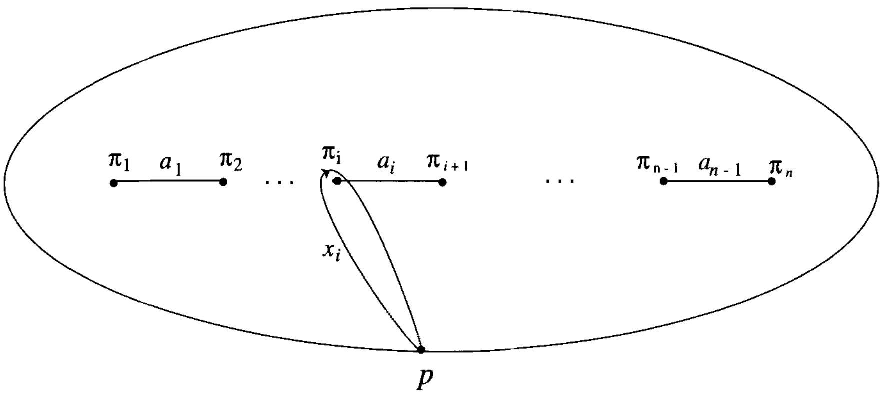

Figure 2.

For $1 \leq i < j \leq n+1$ we let

$$\sigma_{ij} = \sigma_i \sigma_{i+1} \cdots \sigma_{j-2} \sigma_{j-1} \sigma_{j-2}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^{-1} \in A(A_n).$$

Let

$$\Sigma_n = \{\sigma_{ij} \colon 1 \leq i < j \leq n+1\} \subset A(A_n).$$

We note that if $\sigma \in \Sigma_n$, then the arc $a(\sigma)$ can be drawn in $\mathcal{D}_{n+1}$ so that it is completely below the upper half plane. This is how we will always draw the elements of Σ_n. Note that there are $\binom{n+1}{2}$ elements of Σ_n. We will show that these are the special Hurwitz generators.

The following result will show that the set of special Hurwitz generators for the $\mathcal{B}_n$ action on $(t_1, \ldots, t_n)$ in the A_n case is finite. This result thus proves Theorem 1.1 in the A_n case.

Proposition 2.1: *For any $\alpha \in \mathcal{B}_n$ with $\alpha(t_1, \ldots, t_n) = (y_1, \ldots, y_n)$, we have $a(y_i) \in \Sigma_n$ for $1 \leq i \leq n$. Further, if $1 \leq i \neq j \leq n$, then $a(y_i)$ and $a(y_j)$ are* *disjoint (except possibly at their end points) and the union of the arcs in $\mathfrak{A}(\alpha)$ is a tree with vertices the punctures of $\mathcal{D}_{n+1}$.*

Proof: This will be by induction on the length of $\alpha \in \mathcal{B}_n$ as a product of the standard generators $\sigma_i^{\pm 1}$. The case of zero length ($\alpha = id$) is clear.

Now let $\alpha = \sigma_k^\epsilon \beta, \epsilon = \pm 1, 1 \leq k < n$, where $\beta \in \mathcal{B}_n$ has shorter length than α. Let $\beta(t_1, \dots, t_n) = (y_1, \dots, y_n)$, where we may assume inductively that each $a(y_i)$ is an arc below the upper half plane and that they have pair-wise disjoint interiors with $\mathfrak{A}(\beta)$ a tree.

We now see how σ_k^ϵ acts; by (1.1) σ_k^ϵ clearly only effects y_k, y_{k+1}. There are two cases. First assume that the elements y_k, y_{k+1} commute. This is equivalent to requiring the arcs $a(y_k), a(y_{k+1})$ to have disjoint end points as well as disjoint interiors. One possibility is shown in Figure 3 (i) (any other case is similar). Then the effect on $(y_1, \dots, y_n)$ of acting by $\sigma_k^\epsilon, \epsilon = \pm 1$, is to interchange y_k and y_{k+1}. The effect on $\mathfrak{A}(\beta)$ is to interchange the labels $k, k+1$ as indicated in Figure 3 (i). Thus $\mathfrak{A}(\alpha)$ also satisfies the conclusions of Proposition 2.1.

Now assume that the elements y_k, y_{k+1} do not commute. Then by induction this is equivalent to requiring the arcs $a(y_k), a(y_{k+1})$ to have exactly one end point in common and to look like one of the left-hand sides of Figure 3 (ii), (iii), (iv), or any of Figure 3 (vi) (ignore the dashed lines for the moment). Then the effect on $\mathfrak{A}(\beta)$ of acting by σ_k is shown in Figure 3 (ii), (iii), (iv) for these cases (ignore Figure 3 (vi) for the moment) while the effect on $\mathfrak{A}(\beta)$ of acting by σ_k^{-1} on the left-hand side of Figure 3 (ii) is shown in Figure 3 (v). Note that σ_k^2 has the same effect as σ_k^{-1} on the left-hand side of (ii) (and of (iii), (iv)) and so we have described what happens for all cases except those listed in Figure 3 (vi).

Consider further the action of σ_k^ϵ on the left-hand sides of Figure 3 (ii), (iii), (iv), where $1 \leq r < s < t \leq n$. Note that in each case the graph corresponding to $\mathfrak{A}(\alpha)$ is still connected, and since $\mathfrak{A}(\beta)$ is a tree and we have the same number of edges we see that $\mathfrak{A}(\alpha)$ is also a tree. Now by induction the edges of $\mathfrak{A}(\beta)$ do not cross, so that no other edges enter the triangle determined by the edges $a(y_k), a(y_{k+1})$, together with the dashed edge also shown in the left-hand sides of Figure 3 (ii), (iii), (iv). Thus no other edges enter the triangle determined by the images under σ_k^ϵ of these edges together with the dashed edge also shown in the right-hand sides of Figure 3 (ii), (iii), (iv). Thus no edge crosses these image edges and we have proved Proposition 2.1 for these cases.

We conclude the proof of the $A(A_n)$ case by showing that the situations indicated in Figure 3 (vi) cannot occur. We will consider the first case in Figure

3 (vi), the rest being similar.

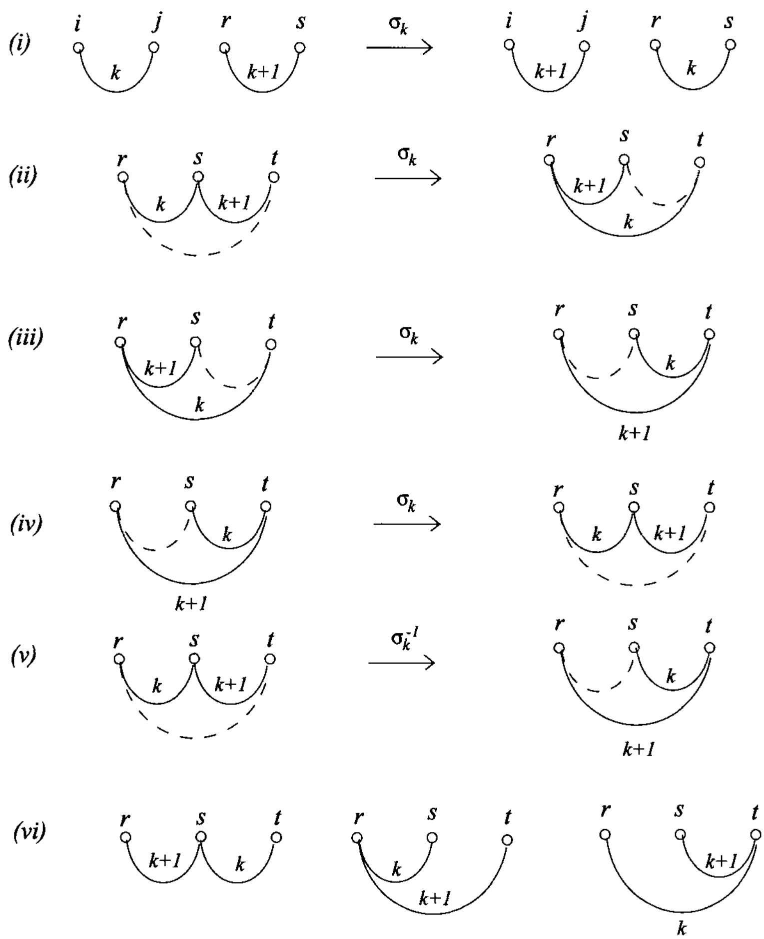

Figure 3.

We recall that a fundamental property of the Hurwitz braid group action is that it preserves the product of the generators: if $\alpha(t_1, \ldots, t_n) = (y_1, \ldots, y_n)$, then $t_1 \cdots t_n = y_1 \cdots y_n$.

Since we can act by $\Pi_{M(A_n)}\colon A(A_n) \to C(A_n) \cong S_{n+1}$ and

$$\Pi = \Pi_{M(A_n)}(\Pi_n(t_1, \ldots, t_n)) = \Pi_{M(A_n)}(t_1 \cdots t_n) = (1, n+1, n, n-1, \ldots, 4, 3, 2),$$

this turns into a result about products of transpositions in S_{n+1}. We have to show that the following is impossible:

(*) let $\beta \in \mathcal{B}_n$, let $1 \leq k < n$, and let $\tau_1, \ldots, \tau_n, \tau_i = \Pi_{M(A_n)}(y_i(\beta))$, be transpositions generating S_{n+1} such that (acting on the right) we have $\tau_1 \cdots \tau_n = (1, n+1, n, n-1, \ldots, 4, 3, 2)$, where $\tau_k = (st), \tau_{k+1} = (rs)$ with $1 \leq r < s < t$.

Proof of (*): We prove this by induction, attaching statement (*) to the inductive hypothesis that we are using to prove Proposition 2.1. The initial cases are trivial and we assume $\alpha = \sigma_k^\epsilon \beta$ etc. Thus we assume that $\mathfrak{A}(\beta)$ is as described in Proposition 2.1 and show that (*) is impossible.

We first note that we may assume that $k = 1$. This is just [Hum; Lemma 2.1 (iii)], which shows how to move any subsequence $y_{i_1}, \ldots, y_{i_u}, 1 \leq i_1 < i_2 \cdots < i_u \leq n$, of $y_1, \ldots, y_n$ into the first u positions by using an action of $\mathcal{B}_n$.

Writing $\tau_1 = (st), \tau_2 = (rs), \tau_3 \cdots \tau_n = \delta$, we then have $t > 1$ and so

$$t - 1 = (t)\tau_1 \cdots \tau_n = (t)(st)(rs)\delta = (r)\delta.$$

But the only way that we can have $(r)\delta = t - 1$ is if there is a path in $\mathfrak{A}(\beta)$ from r to one of r, s, t which does not use the edges $a(y_k), a(y_{k+1})$. But if this is the case, then, since all of the edges of $\mathfrak{A}(\beta)$ are arcs below the x-axis we see that $\mathfrak{A}(\beta)$ is not a tree; a contradiction. Thus the cases shown in Figure 3 (vi) cannot occur and so concludes the proof of (*) and so, by induction, of Proposition 2.1. ∎

Now the finiteness result in Theorem 1.1 for type A_n follows since by the above Proposition we have

$$\mathcal{B}_n(t_1, \ldots, t_n) \subset \Sigma_n^n,$$

where $\mathcal{B}_n(t_1, \ldots, t_n)$ denotes the orbit of $(t_1, \ldots, t_n)$ under the $\mathcal{B}_n$ Hurwitz action.

Recall that $\mathcal{S}(C(A_n))$ denotes the set of conjugacy classes of sequences of n transpositions generating $S_{n+1} \cong C(A_n)$. Let $\bar{\mathcal{S}}(C(A_n))$ denote the set of sequences of n transpositions generating S_{n+1}.

Now if we consider the map $\Pi_{M(A_n)}\colon A(A_n) \to C(A_n)$, then this map induces maps

$$\bar{\Pi}_{M(A_n)}\colon \mathcal{B}_n(t_1, \ldots, t_n) \to \bar{\mathcal{S}}(C(A_n)); \quad \Pi_{M(A_n)}\colon \mathcal{B}_n(t_1, \ldots, t_n) \to \mathcal{S}(C(A_n)).$$

Let $\bar{O}$ and O denote the images of $\bar{\Pi}_M$ and Π_M respectively. For a set X, let $\mathrm{Sym}(X)$ denote the symmetric group of X. Then we have monomorphisms

$$\bar{\rho}\colon \mathrm{Sym}(\mathcal{B}_n(t_1, \ldots, t_n)) \to \mathrm{Sym}(\bar{O}); \quad \rho\colon \mathrm{Sym}(\mathcal{B}_n(t_1, \ldots, t_n)) \to \mathrm{Sym}(O),$$

which when restricted to the image of $\mathcal{B}_n$ in $\mathrm{Sym}(\mathcal{B}_n(t_1,\ldots,t_n))$ give permutation representations of $\mathcal{B}_n$ that we will also denote by $\bar{\rho},\rho$.

We first show that $\bar{\rho}$ is injective. Now for $1\le i\ne j\le n$ we have $\Pi_{M(A_n)}(\sigma_{ij}) = (ij)\in S_{n+1}$ and so the map $\mathcal{B}_n(t_1,\ldots,t_n)\to\Sigma_n^n$ induced by $\Pi_{M(A_n)}$ is injective. Proposition 2.1 shows that the actions of $\mathcal{B}_n$ on $x\in\mathcal{B}_n(t_1,\ldots,t_n)$ and on $\Pi_{M(A_n)}(x)\in\Pi(\Sigma_n^n)$ are permutation isomorphic under $\Pi_{M(A_n)}$. It follows that $\bar{\rho}$ is injective.

The map ρ is onto by [Hum; Theorem 1.2] and from [Hum; Theorem 1.4] we see that O has size $(n+1)^{n-2}$. Now if $\alpha_1,\alpha_2\in\mathcal{B}_n$ satisfy $\Pi_{M(A_n)}(\alpha_1(t_1,\ldots,t_n)) = \Pi_{M(A_n)}(\alpha_2(t_1,\ldots,t_n))$, then $\alpha_1(t_1,\ldots,t_n)$ and $\alpha_2(t_1,\ldots,t_n)$ must be conjugate. But this conjugating element must commute with the product $t_1\cdots t_n$ and so must be a power of that element. (Note that $\Pi=\Pi_{M(A_n)}(t_1\cdots t_n)$ is an $n+1$-cycle in S_{n+1} and that it generates its own centraliser.) But [Bi] the generator of the centre of $\mathcal{B}_n$ acts by conjugating by this element. Thus $\Pi_{M(A_n)}(t_1\cdots t_n)$ has order $n+1$ and so there are exactly $n+1$ conjugates of $(t_1,\ldots,t_n)$ which get sent to the same element. Thus there are $(n+1)\times(n+1)^{n-2}=(n+1)^{n-1}$ elements in the orbit $\mathcal{B}_n(t_1,\ldots,t_n)$. This concludes the proof of the A_n case of Theorem 1.1. ∎

3. The B_n case

The proof for this case is similar to that of the A_n case in that we represent special Hurwitz generators by (unions of disjoint) arcs.

Here we use a way of representing the Artin group $A(B_n)$ as a subgroup of $A(A_{2n-1})\cong\mathcal{B}_{2n}$ that is described in [C] and [SV]. Define the following elements of $A(A_{2n-1})$:

$$Q_1=\sigma_1\sigma_{2n-1},\quad Q_2=\sigma_2\sigma_{2n-2},\quad \ldots,\quad Q_{n-1}=\sigma_{n-1}\sigma_{n+1},\quad Q_n=\sigma_n.$$

Then [C, SV] the group $\langle Q_1,Q_2,\ldots,Q_{n-1},Q_n\rangle$ is isomorphic to $A(B_n) = \langle t_1,\ldots,t_n\rangle$, under the map induced by $Q_i\mapsto t_i$.

We now assume that $\mathcal{D}_{2n}$ is a circular disc centered at $(0,0)$ of radius $n+1$ such that the punctures $\pi_1,\ldots,\pi_{2n}$ of $\mathcal{D}_{2n}$ are the points

$$(-n,0),(-n+1,0),\ldots,(-1,0),(1,0),\ldots,(n-1,0),(n,0),$$

respectively.

We let $I=[-1/10,1/10]\subset\mathcal{D}_{2n}$ denote the interval of the x-axis (see Figure 4 (i)).

Now as in the A_n case we represent each $\sigma_{ij} \in A(A_{2n-1}), 1 \le i < j \le 2n$, by an arc $a(\sigma_{ij}) \subset \mathcal{D}_{2n}$. There are various cases: If $i, j \le n$, then $a(\sigma_{ij})$ will be a semi-circular arc joining the punctures π_i, π_j below the x-axis of $\mathcal{D}_{2n}$. If $i, j > n$, then $a(\sigma_{ij})$ will be a semicircular arc joining the punctures π_i, π_j above the x-axis of $\mathcal{D}_{2n}$. If $i \le n, j > n$, then $a(\sigma_{ij})$ is the union of a semicircle below the x-axis between π_i and $(0,0)$ and a semicircle above the x-axis between $(0,0)$ and π_j; see Figure 4 (i), for example.

We thus represent $Q_i, 1 \le i < n$, by a union of arcs $a(Q_i) = a(\sigma_i) \cup a(\sigma_{2n-i})$, so that Q_i is the product of the positive half-twists corresponding to these arcs. Each $A(B_n)$-conjugate of $Q_i, i < n$, can also be represented by a pair of disjoint arcs. For Q_n we define $a(Q_n) = a(\sigma_n)$ to be an arc which is the union of a semicircle below the x-axis between $(-1,0)$ and $(0,0)$ and a semicircle above the x-axis between $(0,0)$ and $(1,0)$.

Let $\tau = \tau_{2n} \in \mathcal{B}_n$ denote the involutive diffeomorphism of $\mathcal{D}_{2n}$ which rotates by π about $(0,0)$. Then τ_{2n} also corresponds to the permutation $(1,2n)(2,2n-1)\cdots(n,n+1)$ of the punctures.

Now if $\sigma = \sigma_{ij}\sigma_{rs} \in A(B_n)$, then we let $a(\sigma) = a(\sigma_{ij}) \cup a(\sigma_{rs})$. If $a(\sigma)$ is τ_{2n}-invariant, then exactly two of i, j, r, s are less than $n+1$; if these are u, v, then $\{i,j,r,s\} = \{u, v, 2n+1-u, 2n+1-v\}$. In particular, since there are two ways to connect $\{u, v, 2n+1-u, 2n+1-v\}$ with two arcs (such that the two arc components are interchanged by τ_{2n}), there are $2\binom{n}{2} = n^2 - n$ of these τ_{2n}-invariant σs. There are also n of the $\sigma_{i,2n+1-i}$ which are τ_{2n}-invariant. Thus there are n^2 of the arcs of the form $a(\sigma)$ described above which are τ_{2n}-invariant. These $\sigma = \sigma_{ij}\sigma_{2n+1-i,2n+1-j}$ and $\sigma = \sigma_{i,2n+1-i}$ will be the special Hurwitz generators. Let Σ_n denote the set of all these σs.

In the following we will allow the arcs which meet I to move slightly, but never to move entirely off I. We do this so that sets of arcs will then have disjoint interiors.

The $a(Q_i)$ are symmetric relative to τ_{2n}: $a(Q_i)$ satisfies

$$\tau_{2n}(a(Q_i)) = a(Q_i).$$

It easily follows that if $\alpha \in \langle Q_1, \ldots, Q_n\rangle = A(B_n) \subset A(A_{2n-1})$, then

$$\tau_{2n}(\alpha(a(Q_i))) = \alpha(a(Q_i)).$$

Proposition 3.1: *Let $\alpha \in \mathcal{B}_n$ and let $Q_1, \ldots, Q_n \in A(B_n) \subset A(A_{2n-1})$ be the standard generators for $A(B_n)$ as described above. Then we have $\alpha(Q_1, \ldots, Q_n) = (y_1, \ldots, y_n)$ where each of the $a(y_i)$ having two components b_{i1}, b_{i2} satisfies*

$$\tau_{2n}(b_{i1}) = b_{i2}, \quad \tau_{2n}(b_{i2}) = b_{i1},$$

and the interiors of the b_{ij} can be isotoped (in a small neighbourhood of I) to be pair-wise disjoint. Further, each b_{ij} is equal to some $a(\sigma_{uv})$. Also, if the end points of b_{i1}, b_{i2} are j, k and $2n+1-j, 2n+1-k$ (respectively), then $\Pi(y_i) = (j,k)(2n+1-j, 2n+1-k)$.

If $a(y_i)$ has a single component, then it is equal to some $a(\sigma_{u,2n+1-u})$ and is τ_{2n}-invariant. Further, if the end points of $a(y_i)$ are $u, 2n+1-u$, then $\Pi(y_i) = (u, 2n+1-u)$.

In either case, if $a(y_i) = b_1 \cup b_2$ with b_1 below the x-axis and b_2 above the x-axis, then the left-most endpoint of b_1 is the left-most of all the vertices of $a(y_i)$ and the right-most endpoint of b_2 is the right-most of all the vertices of $a(y_i)$.

The set $\Sigma = \Sigma_n$ is the set of special Hurwitz generators.

Proof: Note that there is an implied correspondence (determined by $\Pi_{M(\mathcal{B}_n)}$) between the elements of Σ and the conjugates of $\Pi_{M(\mathcal{B}_n)}(Q_i), i \leq n$. We will be showing that this is respected by the $\mathcal{B}_n$ action.

Since $\tau_{2n}(a(Q_i)) = a(Q_i)$ for all $i \leq n$ we see that $\tau_{2n}Q_i = Q_i\tau_{2n}$.

If $\alpha(Q_1, \ldots, Q_n) = (y_1, \ldots, y_n)$, then each y_i has the form $E_jQ_jE_j^{-1}$ for some $E_j \in \langle Q_1, \ldots, Q_n\rangle$; we then have $a(y_i) = E_j(a(Q_j))$ and so $a(y_i)$ is τ_{2n}-invariant by the above.

The rest of the proof will again be by induction on the length of α, as a word in the standard $\mathcal{B}_n$ generators $\sigma_i^{\pm 1}, i < n$, the case $\alpha = id$ being clear. So assume that $\alpha = \sigma_k^{\epsilon}\beta$, where β has smaller length than α and $\epsilon = \pm 1$. By induction we may assume that $\beta(Q_1, \ldots, Q_n) = (y_1, \ldots, y_n)$, where the arcs of all of the $a(y_i)$ satisfy the conditions of Proposition 3.1.

We now see how σ_k^{ϵ} acts on $\beta(Q_1, \ldots, Q_n)$; again it only effects y_k, y_{k+1}. There are two cases. First assume that the elements y_k, y_{k+1} commute. This is equivalent to requiring that $a(y_k), a(y_{k+1})$ have disjoint end points (since $y_k \neq y_{k+1}$). Then the effect on $(y_1, \ldots, y_n)$ of acting by σ_k^{ϵ} is to interchange y_k and y_{k+1}. The effect on $\mathfrak{A}(\beta)$ is to interchange the labels $k, k+1$. Thus $\mathfrak{A}(\alpha)$ satisfies the conclusions of Proposition 3.1.

Now assume that the elements y_k, y_{k+1} do not commute. Here there are two cases:

(a) $a(y_k), a(y_{k+1})$ both have two components;

(b) one of $a(y_k), a(y_{k+1})$ has one component.

If (a) and $a(y_k) = c_1 \cup c_2, a(y_{k+1}) = d_1 \cup d_2$, where the arc components c_1, d_1 are both below the x-axis and c_2, d_2 are both above the x-axis with further c_1, d_1 and c_2, d_2 each sharing a puncture, then this case follows from the analysis of the $A(A_n)$ case above (see Figure 3 (ii)–(v)) since we just have two disjoint copies of the $A(A_n)$ case, one above and one below the x-axis.

On the other hand, if we have (a) but not the above case, then we will show that we have Figure 4 (i), (ii) or (iii) and for these cases we indicate how σ_k acts in Figure 4. Here X, Y, Z are parts of the picture and $\tau X, \tau Y, \tau Z$ are their images under $\tau = \tau_{2n}$. Note that σ_k has order three on such elements, so the effect of σ_k^{-1} is also apparent from Figure 4. It follows that $\mathfrak{A}(\alpha)$ satisfies Proposition 3.1.

This does (a) if we can show that Figure 4 (i)–(iii) are the only possibilities. We do this after considering (b).

If (b), then we will show that $a(y_k), a(y_{k+1})$ look like one of Figure 5 (i), (ii), (iii), (iv). If this is the case, then the effect of σ_k is indicated in Figure 5, from which we see that σ_k^4 acts as the identity, so that the action of σ_k^{-1} is the same as the action of σ_k^3. One can now easily check that α satisfies the conclusions of Proposition 3.1.

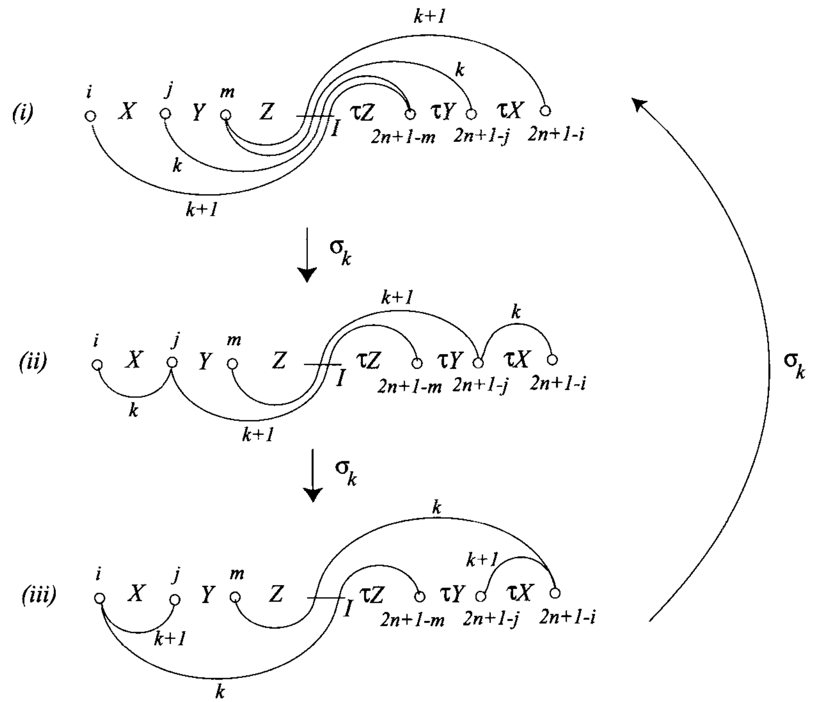

Figure 4.

We thus need to show that $a(y_k), a(y_{k+1})$ look like one of Figure 4 (i), (ii), (iii)

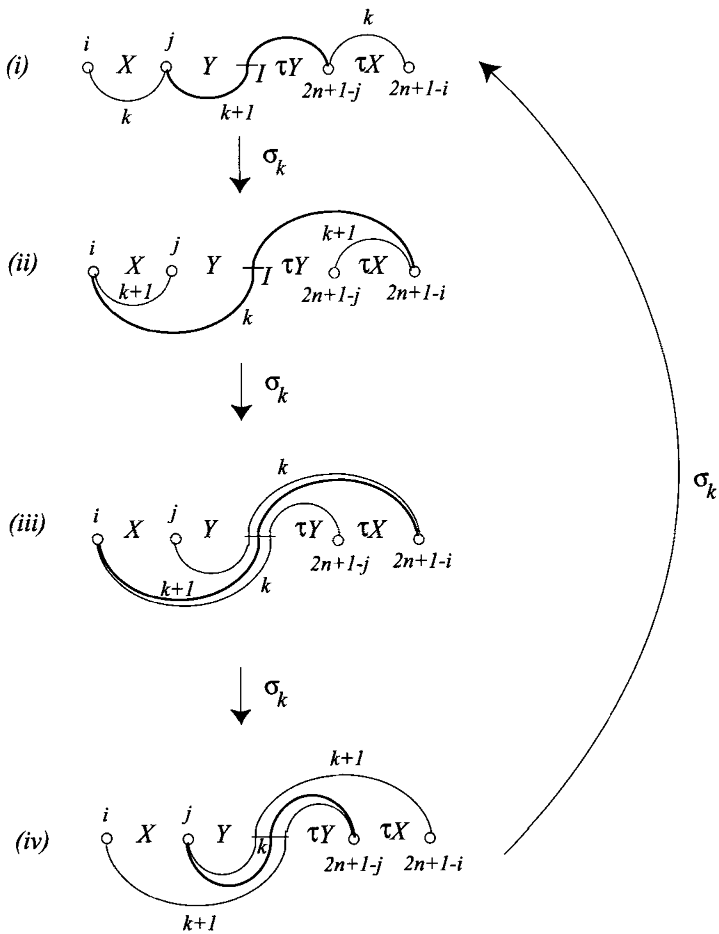

Figure 5.

in case (a) or Figure 5 (i), (ii), (iii), (iv) in case (b). This will be similar to the $A(A_n)$ case since again it turns out that we need only look at the image under $\Pi_{M(B_n)}$: $A(B_n) \to C(B_n)$, which we now describe. Recall that $C(B_n)$ is the group of all *signed permutations* of $\pm 1, \ldots, \pm n$ [GB]. We order these elements as $1, 2, \ldots, n, -n, 1-n, \ldots, -1$, and denote them by $1, 2, \ldots, 2n$ (respectively), so that the generators $s_i = \Pi_{M(B_n)}(t_i)$ are just

$$\begin{aligned} s_1 &= (1,2)(2n-1,2n), \\ s_2 &= (2,3)(2n-2,2n-1), \\ &\ \vdots \\ s_{n-1} &= (n-1,n)(n+1,n+2), \\ s_n &= (n,n+1). \end{aligned}$$

Thus if $\alpha(Q_1,\ldots,Q_n) = (y_1,\ldots,y_n)$, where $a(y_i)$ consists of two arcs c_1, c_2 whose endpoints are $\{i,j\},\{2n+1-i,2n+1-j\}, 1 \leq i \neq j \leq 2n$, then the corresponding permutation is $\Pi(y_i) = (i,j)(2n+1-i,2n+1-j)$. If $a(y_i)$ is a single arc with endpoints $i, 2n+1-i$, then $\Pi(y_i) = (i,2n+1-i)$. Thus the permutation can be directly read off from $a(y_i)$. The product is then

$$s_1 s_2 \cdots s_n = (1,n+1,n+2,\ldots,2n-1,2n,n,n-1,n-2,\ldots,4,3,2).$$

One can check that for case (b), if $a(y_k), a(y_{k+1})$ do not look like the cases shown in Figure 5 (i)–(iv), then they must look like one of Figure 5 (i)–(iv) only with the labels $k, k+1$ interchanged, a similar statement being true for Figure 4 in case (a).

We first note that, as in the A_n case, we may assume that $k=1$ [Hum; Lemma 2.1 (iii)].

Suppose that $a(y_k), a(y_{k+1})$ look like Figure 6, which is Figure 5 (i) with the labels $k(=1), k+1(=2)$ interchanged (any other case in Figure 5 being similar). Here $1 \leq i < j \leq n$.

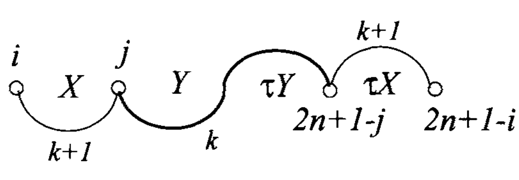

Figure 6.

Then

$$\begin{aligned} z_1 &= \Pi_{M(B_n)}(y_1) = (j,2n+1-j), \\ z_2 &= \Pi_{M(B_n)}(y_2) = (ij)(2n+1-i,2n+1-j) \end{aligned}$$

and so we have

$$\begin{aligned}(1,n+1,n+2,\dots,&2n-1,2n,n,n-1,n-2,\dots,4,3,2)\\ &=(j,2n+1-j)(ij)(2n+1-i,2n+1-j)z_3\cdots z_n\\ &=(i,j,2n+1-i,2n+1-j)z_3\cdots z_n\end{aligned}$$

where $z_i=\Pi_{M(B_n)}(y_i)$. Since $j>i\geq 1$, we have $j-1=(j)\Pi_n$.

Since $(j)z_1z_2=(j)(i,j,2n+1-i,2n+1-j)=2n+1-i$, we see that $(2n+1-i)z_3z_4\cdots z_n=j-1$ and so there must be a path in $\mathfrak{A}(\beta)$ with endpoints $2n+1-i, j-1$, which only contains edges of $a(y_3)\cup a(y_4)\cup\cdots\cup a(y_n)$ and so which does not contain the edges $a(y_1),a(y_2)$. One can see, using the fact that $\mathfrak{A}(\beta)$ satisfies Proposition 3.1, that this path must eventually intersect the interval I and then continue in the lower half plane to $j-1$; in doing so it must meet (at a vertex) the part of $a(y_1)\cup a(y_2)$ which is in the lower half plane. This contradicts the fact that $\mathfrak{A}(\beta)$ is a tree. This concludes considerations of Figure 5 and so of case (b).

For Figure 4 and case (a) we consider, for example, Figure 4 (ii) with the labels $k,k+1$ interchanged. Again one sees that $1<j<n$ and so

$$j-1=(j)z_1z_2z_3\cdots z_n=(2n+1-m)z_3\cdots z_n,$$

showing that there is an edge path in $\mathfrak{A}(\beta)\setminus\{a(y_1),a(y_2)\}$ from $2n+1-m$ to $j-1$ and that any such edge path must meet one of the vertices i,j, thus showing that $\mathfrak{A}(\beta)$ is not a tree. This contradiction shows that this case does not occur and all other cases are similar.

One now sees that Proposition 3.1 is true for α and so the result follows by induction. ∎

Now by Proposition 3.1 we note that there are only a finite number of the y_i that occur as entries in $\alpha(Q_1,\dots,Q_n)$ and so the orbit is finite. Proposition 3.1 also shows that the endpoints of any one of the arcs $a(y_i)$ completely determine that arc and so determines the element y_i. Now there is an epimorphism $\Pi_{M(B_n)}\colon A(B_n)\to C(B_n)$ which the last comments show is injective when restricted to the set of special generators. Thus this map induces an injective map $\Pi_{M(B_n)}\colon \mathcal{B}_n(Q_1,\dots,Q_n)\to C(B_n)^n$, whose image has cardinality n^{n-1} by [Hum]. An argument as in the A_n case shows that the orbit $\mathcal{B}_n(Q_1,\dots,Q_n)$ has size $n\times n^{n-1}=n^n$. This completes the proof of Theorem 1.1 for the B_n case. ∎

4. The D_n case

In this section we prove Theorem 1.1 for $A(D_n)$, this being the most involved case. We again need a geometric way of representing the group $A(D_n)$ and its standard generators. To do this we use a result of Perron and Vannier [PV] which allows one to represent $A(D_n)$ as a subgroup of a mapping class group generated by Dehn twists. Specifically, let $a_1, a_2, \ldots, a_n$ be a set of non-bounding simple closed curves on a surface with boundary which are homologically independent and whose intersections are of type D_n.

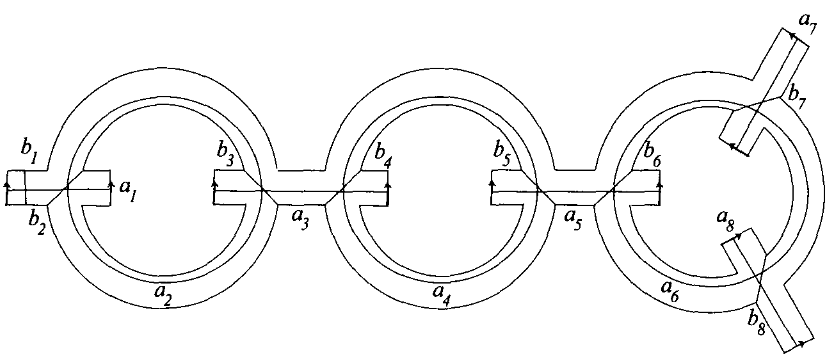

Figure 7.

See Figure 7 for example, where $n = 8$ and we have drawn the curves $a_1, \ldots, a_8$ as if on a genus five surface. The identifications have been indicated showing how to join up some of the curves that look like arcs. We have also indicated arcs $b_2, \ldots, b_n$, one for each intersection point of a pair of curves $a_i, a_{i+1}, 1 \leq i < n$, together with an arc b_1 meeting only a_1 as shown. The arcs b_i have their endpoints on the boundary of the surface. Thus the surface that we have drawn is a tubular neighbourhood N_n of the union $\bigcup_{i=1}^{n} a_i$, having a number of boundary components. Note that cutting the surface along the b_i results in a disc with identifications indicated by the sequence

$$b_1, b_2, \ldots, b_{n-2}, b_{n-1}, b_n, b_1^{-1}, b_2^{-1}, \ldots, b_{n-2}^{-1}, b_n^{-1}, b_{n-1}^{-1},$$

as shown in Figure 8 for the case $n = 8$ again. In Figure 8 we have also indicated orientations to the b_i and the a_i so that $b_i.a_i = 1$ for all $i \leq n$ etc.

For each a_i we let $T(a_i)$ denote the Dehn twist about the curve a_i. By [PV] the subgroup $\langle T(a_1), \ldots, T(a_n)\rangle$ is a faithful representation of $A(D_n)$ under the homomorphism induced by $t_i \mapsto T(a_i)$. Note further that the Dehn twist associated to a simple closed curve does not depend on the orientation of that curve, just on the orientation of the surface. Thus we can freely change orientations of such curves when they are representing Dehn twists, as they will do in what follows.

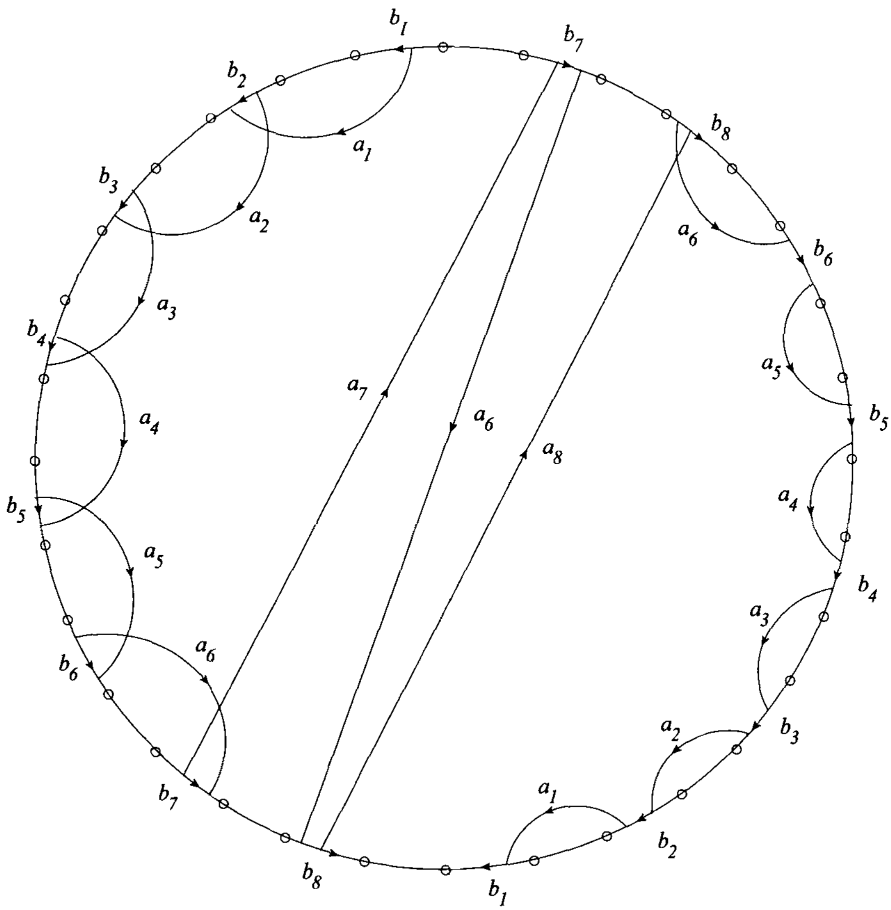

Figure 8.

Now we describe certain *symplectic coordinates* for curves in N_n. Let c be an oriented simple closed curve in N_n and isotope c so that it intersects the b_i minimally. Let $\iota'(b_i, c)$ denote this minimal intersection number. If, for fixed $i \leq n$, all of the local intersections of b_i and c have the same orientation, $\epsilon \in \{\pm 1\}$ (this will always be the case for the curves we consider), then we let $\iota(b_i, c) = \epsilon\iota'(b_i, c)$. Then the symplectic coordinates are

$$\bar{c} = (\iota(b_1, c), \iota(b_2, c), \ldots, \iota(b_n, c)) \in \mathbb{Z}^n.$$

For example, from Figure 8 we have

$$\begin{aligned}
\bar{a}_1 &= (1,-1,0,0,\dots,0),\\
\bar{a}_2 &= (0,1,-1,0,\dots,0),\\
&\vdots\\
\bar{a}_{n-3} &= (0,0,0,\dots,1,-1,0,0),\\
\bar{a}_{n-2} &= (0,0,0,\dots,0,1,-1,-1),\\
\bar{a}_{n-1} &= (0,0,0,\dots,0,1,0),\\
\bar{a}_n &= (0,0,0,0,\dots,0,0,1).
\end{aligned}$$

If T is the Dehn twist about a curve c, then we will also let $\bar{T}$ denote the symplectic coordinates of the curve c. Note that $N_n \setminus \bigcup_{i=1}^{n} b_i$ is a disc and so there are only finitely many simple closed curves c whose coordinates are in $\{0,\pm 1\}^n$.

We next give a list of vectors which we will later show are the symplectic coordinates of the special Hurwitz generators: Let $\mathcal{C}_n$ denote the set of curves having the following symplectic coordinates:

$$\begin{aligned}
&\text{(i) } (0,\dots,0,1,0,\dots,0),\\
&\text{(ii) } (0,\dots,0,1,0,\dots,0,-1,0,\dots,0),\\
&\text{(iii) } (0,\dots,0,1,0,\dots,0,1,0,\dots,0,-1,-1),\\
&\text{(iv) } (0,\dots,0,1,0,\dots,0,-1,-1)
\end{aligned} \tag{4.1}$$

(except that $(0,\dots,0,1,-1)$ is not allowed in the second type listed). Let $\mathcal{V}_n$ denote the above set of vectors.

First note that of the types listed above we have

$$n + \left(\binom{n}{2} - 1\right) + \binom{n-2}{2} + (n-2) = n(n-1)$$

total vectors. Thus $\mathcal{V}_n$ has $n(n-1)$ vectors.

Now it is easy to check that each vector in $\mathcal{V}_n$ determines a unique simple closed curve in N_n. Thus $\mathcal{C}_n$ is well-defined. Since all curves are unoriented, these coordinates are determined only up to a common sign.

Let $(Q_1,\dots,Q_n), Q_i = T(a_i)$, be the standard generators for $A(D_n)$ and $\Pi_{M(B_n)}\colon A(D_n) \to C(D_n)$ the projection; specifically we have the following action on the generators:

$$\begin{aligned}
\Pi_{M(B_n)}(Q_i) &= (i,i+1)(2n+1-i,2n-i), \quad \text{for } 1 \le i \le n-1,\\
\Pi_{M(B_n)}(Q_n) &= (n-1,n+1)(n+2,n).
\end{aligned}$$

We need to specify one more bijection, namely $\Psi\colon \mathcal{V}_n \to \mathcal{P}_n$, where $\mathcal{P}_n$ is the set of elements of $C(D_n)$ that occur as entries in the $\mathcal{B}_n$ orbit of $(\Pi_{M(B_n)}(Q_1),\ldots,\Pi_{M(B_n)}(Q_n))$ (see [Hum]), or alternatively, $\mathcal{P}_n$ is the image of Ψ — see (4.2) below. Let e_i denote the ith unit vector. Then Ψ is defined as follows:

(4.2)

(i) $\Psi(e_i) = (i, n+2)(n-1, 2n+1-i)$ if $1 \le i \le n-2$;

(i) $\Psi(e_{n-1}) = (n-1, n)(n+1, n+2)$;

(i) $\Psi(e_n) = (n-1, n+1)(n, n+2)$;

(ii) $\Psi(e_i - e_j) = (i, j)(2n+1-i, 2n+1-j)$ if $1 \le i < j \ne n-1$;

(ii) $\Psi(e_i - e_{n-1}) = (i, n+1)(2n+1-i, n)$;

(iii) $\Psi(e_i + e_j - e_{n-1} - e_n) = (i, 2n+1-j)(j, 2n+1-i)$ if $1 \le i < j \ne n-2$;

(iv) $\Psi(e_i - e_{n-1} - e_n) = (i, n-1)(2n+1-i, n+2)$.

We will let $\gamma(v)$ denote the unoriented simple closed curve having symplectic coordinates v. In all cases considered this will be unambiguous.

The method of proof is as follows: we have the sets $\mathcal{C}_n, \mathcal{V}_n, \mathcal{P}_n$ and bijections between them. We will show that the $\mathcal{B}_n$ action respects these maps: if $\alpha \in \mathcal{B}_n$ and $\alpha(s_1,\ldots,s_n) = (z_1,\ldots,z_n), \alpha(Q_1,\ldots,Q_n) = (y_1,\ldots,y_n)$, then $\Psi^{-1}(z_i) = \bar{y}_i$ for all $1 \le i \le n$ and $\gamma(y_i) \in \mathcal{C}_n$.

We will have constant need to find dot (symplectic) products of the vectors $\bar{a}(y_i)$; it is easily checked that these are obtained using the following $n \times n$ anti-symmetric matrix form (see Figure 8 to see that this is the correct matrix):

$$\mathcal{I}_n = \begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & \ldots & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & \ldots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \ldots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \ldots & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & \ldots & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & \ldots & -1 & 0 & 0 \end{pmatrix}.$$

So the dot product of two row vectors v_1, v_2 will be $v_1.v_2 = v_1 \mathcal{I}_n v_2^T$, where T denotes transpose.

We note that these dot products (which are called algebraic intersection numbers) of the symplectic coordinates of elements of $\mathcal{C}_n$ are also equal to the geometric intersection numbers of the simple closed curves that they represent (when their absolute values are taken). Part of what we will show below for the proof will be: let $\alpha \in \mathcal{B}_n$ where $\alpha(Q_1, \ldots, Q_n) = (y_1, \ldots, y_n)$. Then $\sigma_1(y_1, \ldots, y_n) = (y_1 y_2 y_1^{-1}, y_1, y_3, \ldots, y_n)$ where

$$\bar{a}(y_1 y_2 y_1^{-1}) = \bar{a}(y_2) + (\bar{a}(y_1).\bar{a}(y_2))\bar{a}(y_1) \tag{4.3}$$

is an element of $\mathcal{V}_n$. The above formula is obtained as follows: if c, d are simple closed curves on a closed surface Σ of genus g, then we let $[c], [d] \in H_1(\Sigma; \mathbb{Z})$ denote their homology classes. The action of the mapping class group $\mathrm{Map}(\Sigma)$ on Σ gives rise to a symplectic representation

$$\mathrm{Map}(\Sigma) \to \mathrm{Aut}(H_1(\Sigma; \mathbb{Z})) \cong \mathrm{Sp}(2g, \mathbb{Z}),$$

the action of the Dehn twist $T(c)$ associated to c being given by

$$T(c)([d]) = [d] + (c.d)[c],$$

where $c.d$ is the algebraic intersection number of the curves; see [MKS].

Proposition 4.1: *Let $\alpha \in \mathcal{B}_n$ and suppose that $\alpha(Q_1, \ldots, Q_n) = (y_1, \ldots, y_n)$. Then each simple closed curve $\gamma(\bar{y}_i)$ meets each arc b_j at most once. Each $\bar{a}(y_i)$ is in $\mathcal{V}_n$ and each $\gamma(\bar{y}_i)$ is in $\mathcal{C}_n$. Further, if $\alpha(s_1, \ldots, s_n) = (z_1, \ldots, z_n)$, then $\Psi^{-1}(z_i) = \bar{y}_i$ for all $1 \le i \le n$.*

Proof: The proof will be by induction on the length of α as a product of the standard generators and their inverses, the case $\alpha = id$ being obvious. So suppose that $\alpha = \sigma_k^{\epsilon}\beta \in \mathcal{B}_n$, where β has shorter length than the length of α, so that the result holds for β. Let $\beta(Q_1, \ldots, Q_n) = (y_1, \ldots, y_n)$ and let $z_i = \Pi_{M(D_n)}(y_i)$. Let $s_i = \Pi_{M(D_n)}(Q_i) \in C(D_n)$.

We will again make use of a graph $\mathfrak{A}(\alpha)$, although this time it will not be realised in $\mathcal{D}_{2n}$. Suppose that $\alpha(s_1, \ldots, s_n) = (z_1, \ldots, z_n)$; then $\mathfrak{A}(\alpha)$ is defined to have vertices $1, \ldots, 2n$ and for each $z_i = (r, s)(2n+1-r, 2n+1-s)$ we have two edges between r, s and $2n+1-r, 2n+1-s$ (respectively), where $1 \le r \ne s \le 2n$ and $r + s \ne 2n+1$. Let $\tau_{2n} \in C(D_n)$ be the involution $(1, 2n)(2, 2n-1) \cdots (n, n+1)$. The permutation τ_{2n} will also act on the graphs $\mathfrak{A}(\alpha)$ by permuting vertices. We will let E_i denote the union of the two edges of $\mathfrak{A}(\alpha)$ having label i.

Lemma 4.2: *The graph $\mathfrak{A}(\alpha)$ is connected, invariant under τ_{2n} and has a single circuit $\Gamma(\alpha)$ of even length which contains the vertices $n, n+1$ at diametrically opposed points of this circuit. This circuit is also τ_{2n}-invariant and the effect of τ_{2n} is to rotate the circuit.*

Also, $\Gamma(\alpha) = Y_1 \cup Y_2$ where $Y_1 \cap Y_2 = \{n, n+1\}, \tau_{2n}(Y_1) = Y_2$, where the edges of Y_1 from n to $n+1$ have labels which are strictly increasing and the edges of Y_2 from $n+1$ to n have labels which are strictly increasing.

Proof: Since $C(D_n)$ acts transitively on $\{1, \dots, 2n\}$ we see that the graph $\mathfrak{A}(\alpha)$ is connected. Since τ_{2n} commutes with each generator $(i,j)(2n+1-i, 2n+1-j)$ we see that $\mathfrak{A}(\alpha)$ is τ_{2n}-invariant. Note that each z_i is determined by either of the two arcs of $E_i \subset \mathfrak{A}(\beta)$. Thus the arcs in $E_1, E_2, \dots, E_n$ are distinct, there being $2n$ of them. That there is a single circuit now follows from the fact that $\mathfrak{A}(\alpha)$ is a connected graph with $2n$ vertices and $2n$ edges.

Since the product $z_1 \cdots z_n$ is invariant under the $\mathcal{B}_n$ action and is equal to

$$\Pi_n = (1, n+2, \dots, 2n, n-1, \dots, 3, 2)(n, n+1),$$

we see that the edges along one half of $\Gamma(\alpha)$ between n and $n+1$ must have increasing labels. This concludes the proof of Lemma 4.2. ∎

For the proof of Proposition 4.1 we again consider cases. We will have the following conventions throughout:

$$\alpha = \sigma_k \beta \in \mathcal{B}_n, \quad \alpha(Q_1, \dots, Q_n) = (y_1, \dots, y_n), \quad z_i = \Pi_{M(D_n)}(y_i).$$

First assume that y_k, y_{k+1} commute. Then z_k, z_{k+1} commute and the effect of σ_k^ϵ on $(y_1, \dots, y_n)$ and $(z_1, \dots, z_n)$ is just to interchange y_k, y_{k+1} and z_k, z_{k+1} (respectively).

As in the A_n and B_n cases we may assume that $k = 1$. It will also suffice to assume that we are acting by σ_k^1.

The proof now consists of checking ten cases, depending on which of the four types the vectors $\bar{y}_1, \bar{y}_2$ are. We will do the first four of the ten cases (the cases involving a vector $\bar{y}_i$ of type (i)), and then show how the remaining cases follow from these.

Case (i), (i): Here we first assume that $\bar{a}(y_1) = e_j, \bar{a}(y_2) = e_i$ where $i \leq j < n-1$. First note that $i \neq j$, since $z_1, \dots, z_n$ is a minimal set of 'reflection' generators for $C(D_n)$. Thus $i < j < n-1$ and so $\bar{a}(y_1).\bar{a}(y_2) = e_j.e_i = -1$. Now $z_i = \Psi(\bar{y}_i)$, so that $z_1 = (j, n+2)(2n+1-j, n-1), z_2 = (i, n+1)(2n+1-i, n-1)$. If $\bar{a}(y_1) = e_j, \bar{a}(y_2) = e_i$, then by (4.3) the coordinates of $\sigma_1(y_1, y_2, \dots)$ are $(\bar{a}(y_1 y_2 y_1^{-1}), e_j, \dots) = (e_i - e_j, e_j, \dots)$; we note that $e_i - e_j \in \mathcal{V}_n$ (as required), that the simple closed curves $\gamma(e_i), \gamma(e_j)$ meet in a single point and that the curve $T(\gamma(e_j))(\gamma(e_i))$ has symplectic coordinates $e_i - e_j$. Thus this situation works.

We now show that the case $\bar{a}(y_1) = e_i, \bar{a}(y_2) = e_j$ where $i \le j < n-1$, so that $z_1 = (i, n+1)(2n+1-i, n-1), z_2 = (j, n+2)(2n+1-j, n-1)$, is not possible. This will follow from a more general result:

LEMMA 4.3: *If $\beta \in \mathcal{B}_n$ and $\beta(s_1, \ldots, s_n) = (z_1, \ldots, z_n)$, then for $1 \le k < n$ we cannot have either of the following three cases:*

(1) $z_k = (i, 2n+1-u)(2n+1-i, u); z_{k+1} = (j, 2n+1-u)(2n+1-j, u)$,
(2) $z_k = (i, j)(2n+1-i, 2n+1-j); z_{k+1} = (i, 2n+1-u)(2n+1-i, u)$,
(3) $z_k = (j, 2n+1-u)(2n+1-j, u); z_{k+1} = (i, j)(2n+1-i, 2n+1-j)$,
where $i < j < n, i < u < n, u \neq j$.

Proof: As in the above we may assume that $k = 1$. We will first show that (1) is not possible.

The proof will be accomplished by showing that

$$\begin{aligned}(i, 2n+1-u)(2n+1-i, u) \times (j, 2n+1-u)(2n+1-j, u) \times z_3 \cdots z_n \\ \neq \Pi_n = (1, n+2, n+3, \ldots, 2n, n-1, n-2, \ldots, 3, 2)(n, n+1),\end{aligned}$$

where each z_i has the form $(r, s)(2n+1-r, 2n+1-s)$ and $\mathfrak{A}(\beta)$ satisfies the conclusions of Lemma 4.2. Since the endpoints of E_1, E_2 are $i, j, 2n+1-u \neq n, n+1$ we see from Lemma 4.2 that E_1, E_2 are not edges of $\Gamma(\beta)$. It follows from Lemma 4.2 that we have one of the cases shown in Figure 9 (i), (ii), (iii), each of which we now consider.

Suppose we have Figure 9 (i). Here we write each $E_i = E_i^{(1)} \cup E_i^{(2)}$ as a union of two edges, as shown in Figure 9. Then

$$\begin{aligned}2n+2-u =& (2n+1-u)\Pi_n \\ =& (2n+1-u)(i, 2n+1-u)(2n+1-i, u) \\ & \times (j, 2n+1-u)(2n+1-j, u) z_3 \cdots z_n \\ =& (i) z_3 \cdots z_n,\end{aligned}$$

and so the vertex $2n+2-u$ belongs to a component of the closure of $\mathfrak{A}(\beta) \setminus E_1^{(1)}$ which does not contain $\Gamma(\beta)$; see Figure 9 (iv). Now act by Π_n again. Since

$$2n+3-u = (2n+2-u)\Pi_n = (2n+2-u) z_3 \cdots z_n$$

we see that the vertex $2n+3-u$ also belongs to a component of the closure of $\mathfrak{A}(\beta) \setminus E_1^{(1)}$ which does not contain $\Gamma(\beta)$. Note that $i < u$ implies $2n+1-u < 2n+1-i$. We now repeat this argument, thus showing that all of the vertices labeled $2n+3-u, 2n+4-u, \ldots, 2n+1-i$ belong to a component of the closure of $\mathfrak{A}(\beta) \setminus E_1^{(1)}$ which does not contain $\Gamma(\beta)$. However, this contradicts the fact that the vertex $2n+1-i$ is in the component of the closure of $\mathfrak{A}(\beta) \setminus E_1^{(1)}$ which does contain $\Gamma(\beta)$ (see Figure 9 (i)). This shows that the situation in Figure 9 (i) is not possible.

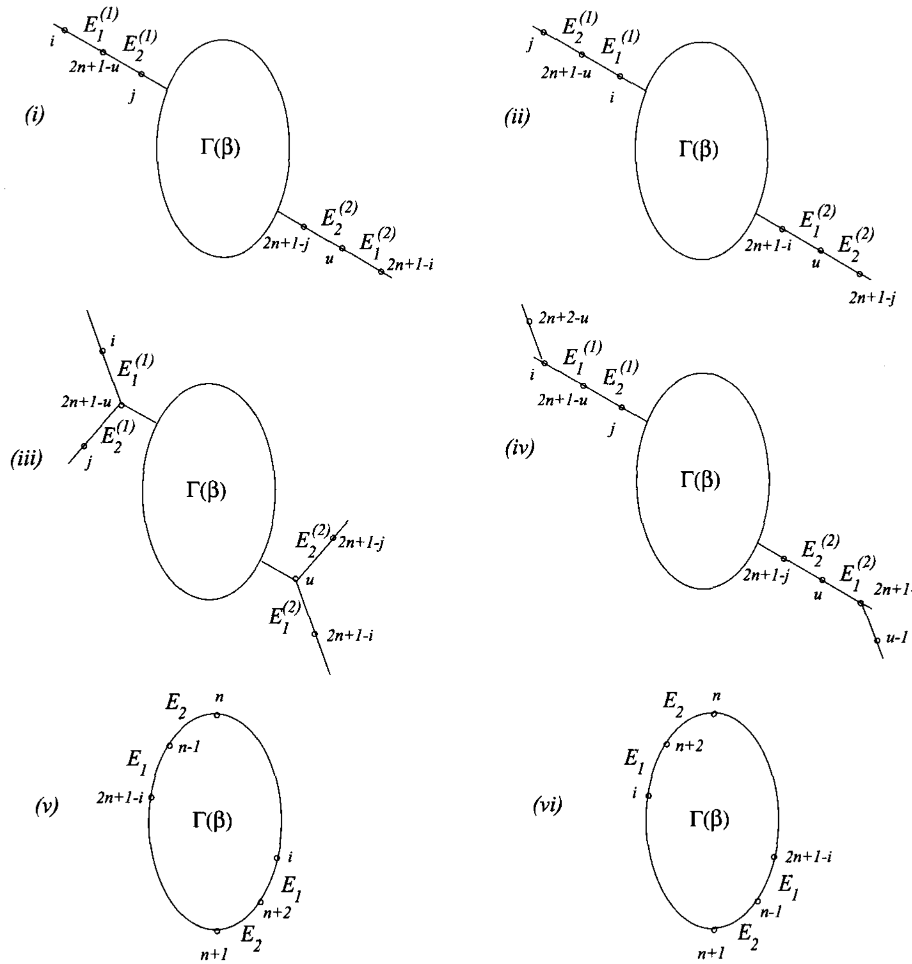

Figure 9.

For Figure 9 (ii) we consider a similar argument, only we look at the image of j. Then, since $j > i \geq 1$, it follows that

$$j - 1 = (j)\Pi_n = (2n + 1 - u)z_3 \cdots z_n$$

is in a component of the closure of $\mathfrak{A}(\beta) \setminus E_1^{(1)}$ which does not contain $\Gamma(\beta)$.

We act again by Π_n, to see that $j-2$ is also in a component of the closure of $\mathfrak{A}(\beta) \setminus E_1^{(1)}$ which does not contain $\Gamma(\beta)$. Continuing in this way shows that (since $i < j$) the vertex labeled i is in a component of the closure of $\mathfrak{A}(\beta) \setminus E_1^{(1)}$ which does not contain $\Gamma(\beta)$. This is a contradiction (see Figure 9 (ii)).

For Figure 9 (iii) we consider the image of i. We have $(i)\Pi_n = i-1$ if $i > 1$. It follows that the vertex labeled $i-1$ is in a component of the closure of $\mathfrak{A}(\beta)\setminus E_2^{(1)}$ which does not contain the vertex $2n+1-u$. Now $i < j$ and so $i-1 < j$. It follows that acting by Π_n again we see that $i-2$ is also in a component of the closure of $\mathfrak{A}(\beta) \setminus E_2^{(1)}$ which does not contain the vertex $2n+1-u$. Repeating as many times as necessary we see that 1 and then $n+2 = (1)\Pi_n, n+3, \dots, 2n+1-u$ are in a component of the closure of $\mathfrak{A}(\beta) \setminus E_2^{(1)}$ which does not contain the vertex $2n+1-u$. This is a contradiction. This concludes the proof of Lemma 4.3 (1).

Now note that if $(z_1,z_2,\dots,z_n)$ is not possible, then neither is $\sigma_1(z_1,z_2,\dots,z_n)$, or $\sigma_1^{-1}(z_1,z_2,\dots,z_n)$; one checks that these cases show that (2) and (3) of Lemma 4.3 are also not possible. ∎

To show that $z_1 = (i,n+1)(2n+1-i,n-1), z_2 = (j,n+2)(2n+1-j,n-1)$, $1 \le i < j < n-1$, is not possible we apply (1) of the above result with $u = n-1$. This concludes the proof of all (i) (i) cases where $j < n-1$.

Next suppose that $i < j = n-1$. There are two cases to consider: (a) $\bar{a}(y_2) = e_{n-1}, \bar{a}(y_1) = e_i$; or (b) $\bar{a}(y_2) = e_i, \bar{a}(y_1) = e_{n-1}$. If we have (a), then $z_2 = (n-1,n)(n+1,n+2), z_1 = (i,n+2)(2n+1-i,n-1)$ and we need to show that $z_1z_2z_3\cdots z_n \neq \Pi_n$. But in this case E_2 has n and $n+1$ as vertices and $E_1 \cap E_2 \neq \emptyset$; it follows from Lemma 4.2 that E_1, E_2 must be edges of $\Gamma(\beta)$ and so we have the situation shown in Figure 9 (v). This contradicts Lemma 4.2 unless $i = n$, which is also not allowed since $i < j = n-1 < n$. Thus (a) is impossible.

If we have (b), then we have

$$z_1 = (n-1,n)(n+1,n+2), \quad z_2 = (i,n+2)(2n+1-i,n-1),$$
$$\bar{a}(y_1) = e_{n-1}, \bar{a}(y_2) = e_i,$$

so that $\bar{a}(y_1y_2y_1^{-1}) = e_i + (e_{n-1}.e_i)e_{n-1} = e_i - e_{n-1} \in \mathcal{V}_n$. One now checks that the curves $\gamma(\bar{y}_1), \gamma(\bar{y}_2)$ meet in a single point and that the symplectic coordinates of $T(\gamma(\bar{y}_1))(\gamma(\bar{y}_2))$ are $e_i - e_{n-1}$ as required. We also have $z_1z_2z_1 = (i,n+1)(2n+1-i,n) = \Psi(e_i - e_{n-1})$. This concludes case (b) and so the case $j = n-1$.

Now if $j = n$ and $z_2 = (n-1,n+1)(n,n+2), z_1 = (i,n+2)(2n+1-i,n-1)$

then we again need to show that $z_1z_2z_3\cdots z_n \neq \Pi_n$. This time Lemma 4.2 forces the situation shown in Figure 9 (vi), which again gives a contradiction.

Lastly, if $j = n$ and $z_1 = (n-1, n+1)(n, n+2), z_2 = (i, n+2)(2n+1-i, n-1)$ then $\bar{a}(y_1) = e_n, \bar{a}(y_2) = e_i$, from which we have $\bar{a}(y_1y_2y_1^{-1}) = e_i + (e_n.e_i)e_n = e_i - e_n \in \mathcal{V}_n$ and $z_1z_2z_1 = (i,n)(2n+1-i, n+1) = \Psi(e_i - e_n)$ as required.

This concludes the proof of case (i), (i).

CASE (i), (ii): Here we first assume that $\bar{a}(y_1) = e_i, \bar{a}(y_2) = e_j - e_k$, where $1 \leq j < k \neq n-1$, and $(j,k) \neq (n-1, n)$.

The first situation to consider is where $i < j < k$. Note that in this case we have $e_i.(e_j - e_k) = 0$ and

$$z_1 = (i, n+2)(2n+1-i, n-1), \quad z_2 = (j,k)(2n+1-j, 2n+1-k).$$

Note that these two permutations commute and also that the effect of σ_1 on y_1, y_2 is to interchange them since the simple closed curves $\gamma(\bar{y}_1), \gamma(\bar{y}_2)$ are disjoint. Thus this case works and the case $\bar{a}(y_2) = e_i, \bar{a}(y_1) = e_j - e_k, i < j < k$, follows similarly.

Consider the case where $\bar{a}(y_1) = e_i, \bar{a}(y_2) = e_j - e_k$ and $i = j$. Then we have $e_i.(e_i - e_k) = -1$. Now if $\bar{a}(y_1) = e_i, \bar{a}(y_2) = e_i - e_k$, then $\bar{a}(y_1y_2y_1^{-1}) = (e_i - e_k) - e_i = -e_k$, which (up to sign) is in $\mathcal{V}_n$, as required. Further, one checks that the curves $\gamma(\bar{y}_1), \gamma(\bar{y}_2)$ corresponding to y_1, y_2 meet in a single point and that $\bar{a}(T(\gamma(\bar{y}_1))(\gamma(\bar{y}_2))) = e_k$, as required. Lastly, we see that

$$\begin{aligned} z_1z_2z_1^{-1} =& (i, n+2)(2n+1-i, n-1)(i,k)(2n+1-i, 2n+1-k) \\ & \times (i, n+2)(2n+1-i, n-1) \\ =& (k, n+2, 2n+1-k, n-1), \end{aligned}$$

as required for the bijective correspondence.

Continuing with the case where $i = j$ we now show that we cannot have $\bar{a}(y_1) = e_i - e_k, \bar{a}(y_2) = e_i$. Here we would have

$$z_1 = (i,k)(2n+1-i, 2n+1-k), \quad z_2 = (i, n+2)(2n+1-i, n-1).$$

That this never happens is a special case of Lemma 4.3 (2) (with $u = n-1$ and $j = k$).

This concludes the discussion of the cases where $i = j$ in case (i), (ii).

Next consider the case $j < i < k$. Then $e_i.(e_j - e_k) = -2$ and we show that this case cannot occur. Here we would have $z_1 = (j,k)(2n+1-j, 2n+1-k), z_2 = (i, n+2)(2n+1-i, n-1)$ or the other way around; since these permutations commute it does not matter which we consider. That this case does not occur will follow from

LEMMA 4.4: *If $\alpha \in \mathcal{B}_n$ and $\alpha(Q_1, \ldots, Q_n) = (y_1, \ldots, y_n)$, then for $1 \leq k < n$ we cannot have $z_k = (j,m)(2n+1-j, 2n+1-m)$, $z_{k+1} = (i, 2n+1-u)(2n+1-i, u)$ (or the other way around) where $j < i < m < n, 1 \leq u < n, i < u \neq m$.*

Proof: As in the above we may assume that $k = 1$.

Note that from the conditions on i, j, m, u we see that neither of E_1, E_2 have $n, n+1$ as vertices and so the edges of E_1, E_2 are not edges of $\Gamma(\beta)$. Thus $\mathfrak{A}(\beta)$ looks like one of Figure 10 (i), (ii).

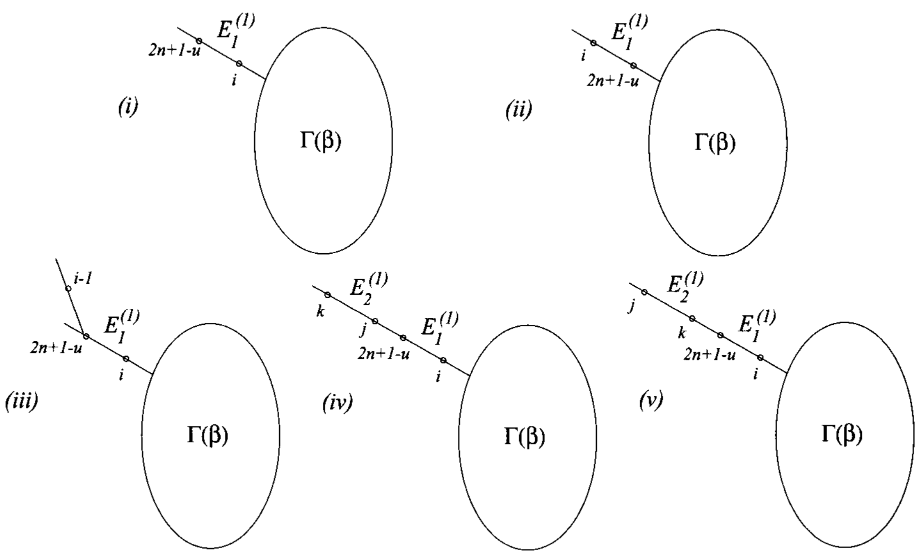

Figure 10.

If we have Figure 10 (i), then $1 \leq j < i$ and $i-1 = (i)\Pi_n = (2n+1-u)z_3 \cdots z_n$ shows that the vertex $i-1$ is in a component of $\mathfrak{A}(\beta) \setminus E_1^{(1)}$ which does not contain $\Gamma(\beta)$; see Figure 10 (iii). Applying Π_n again we see that $i-2$ is in a component of $\mathfrak{A}(\beta) \setminus E_1^{(1)}$ which does not contain $\Gamma(\beta)$. Continuing, we see that j is in a component of $\mathfrak{A}(\beta) \setminus E_1^{(1)}$ which does not contain $\Gamma(\beta)$. Thus we either have the situation shown in Figure 10 (iv) or that in (v). If we have (iv), then we similarly show that $j-1$ is in a component of $\mathfrak{A}(\beta) \setminus E_2^{(1)}$ which does not contain $\Gamma(\beta)$. We continue applying Π to observe that $j-2, j-3, \ldots, 2, 1$ are all in a component of $\mathfrak{A}(\beta) \setminus E_2^{(1)}$ which does not contain $\Gamma(\beta)$ and then that

$n+2, n+3, \ldots, 2n+1-k$ are all in a component of $\mathfrak{A}(\beta) \setminus E_2^{(1)}$ which does not contain $\Gamma(\beta)$. This is a contradiction, since $2n+1-k = \tau_{2n}(k)$ is certainly in a component of $\mathfrak{A}(\beta) \setminus E_2^{(1)}$ which does contain $\Gamma(\beta)$.

The case of Figure 10 (v) is similar: we consider $k-1 = (k)\Pi_n$, seeing that $k-1$ is in a component of $\mathfrak{A}(\beta) \setminus E_2^{(1)}$ which does not contain $\Gamma(\beta)$. Then it follows that $k-2, k-3, \ldots, i+1, i$ are all in a component of $\mathfrak{A}(\beta) \setminus E_2^{(1)}$ which does not contain $\Gamma(\beta)$, this being a contradiction. This concludes the discussion of Figure 10 (i).

If we have Figure 10 (ii), then, by considering the image of $2n+1-u$, it similarly follows that $2n+2-u$ is in a component of $\mathfrak{A}(\beta) \setminus E_1^{(1)}$ which does not contain $\Gamma(\beta)$; and then that $2n+3-u, \ldots, 2n+1-i$ are all in a component of $\mathfrak{A}(\beta) \setminus E_1^{(1)}$ which does not contain $\Gamma(\beta)$. This contradiction concludes the proof of Lemma 4.4 and so of the case $j < i < k$ (where we have $u = n-1$). ∎

Now assume that $i = k$; as in the above this case amounts to showing that we cannot have $z_1 = (k, n+2)(2n+1-k, n-1), z_2 = (j, k)(2n+1-j, 2n+1-k)$, and the proof of this follows the same lines as that in the (i), (i) case where we used Figure 9.

Lastly, the case $i > k$ is similar to the case $i < j$. This concludes all of the (i), (ii) cases except for the situation where $k = n-1$.

If $k = n-1$, then the corresponding permutation is $(j, n+1)(n, 2n+1-j)$, so that the corresponding edge of $\mathfrak{A}(\beta)$ is an edge of $\Gamma(\beta)$. One now easily checks this case using diagrams similar to Figure 9 (v), (vi). This concludes the case (i), (ii).

Case (i), (iii): Here we assume that $\bar{a}(y_1) = e_i, \bar{a}(y_2) = e_j + e_k - e_{n-1} - e_n$ where $1 \leq j < k < n-1$; then we have $z_1 = (i, n+2)(2n+1-i, n-1), z_2 = (j, 2n+1-k)(k, 2n+1-j)$.

Consider first the situation where $i < j$. Then $e_i.(e_j + e_k - e_{n-1} - e_n) = 0$ and z_1, z_2 commute. One also checks that $\gamma(\bar{y}_1), \gamma(\bar{y}_2)$ have zero geometric intersection number. Thus all that happens in this case is that the labels $1, 2$ are interchanged. This does this case.

Next consider where $i = j$. Here $e_j.(e_j + e_k - e_{n-1} - e_n) = -1$ and if $z_1 = (j, n+2)(2n+1-j, n-1), z_2 = (j, 2n+1-k)(k, 2n+1-j)$, then one checks that $\bar{a}(y_1 y_2 y_1^{-1}) = e_k - e_{n-1} - e_n \in \mathcal{V}_n$; that $\gamma(y_1)$ meets $\gamma(y_2)$ in a single point; that $\overline{(T(\gamma(y_1))(\gamma(y_2)))} = e_k - e_{n-1} - e_n$; that $z_1 z_2 z_1 = \Psi(e_k - e_{n-1} - e_n)$ and so that this case works. So we now have to show that the situation $z_1 = (j, 2n+1-k)(k, 2n+1-j), z_2 = (j, n+2)(2n+1-j, n-1)$, is not possible.

LEMMA 4.5: *If $\alpha \in \mathcal{B}_n$ and $\alpha(Q_1,\dots,Q_n) = (y_1,\dots,y_n)$, then for $1 \le k < n$ we cannot have*

(1) $z_k = (j, 2n+1-k)(k, 2n+1-j), z_{k+1} = (j, 2n+1-u)(2n+1-j, u)$,

(2) $z_k = (k,u)(2n+1-k, 2n+1-u); z_{k+1} = (j, 2n+1-k)(k, 2n+1-j)$,

(3) $z_k = (j, 2n+1-u)(u, 2n+1-j); z_{k+1} = (k,u)(2n+1-k, 2n+1-u)$,

where $j < k < n, 1 \le u < n, k < u$.

Proof: As in the above we may assume that $k = 1$. We first do (1).

Note that neither of E_1, E_2 have $n, n+1$ as vertices and so E_1, E_2 are not edges of $\Gamma(\alpha)$. Thus $\mathfrak{A}(\alpha)$ looks like one of Figure 11 (i), (ii), (iii).

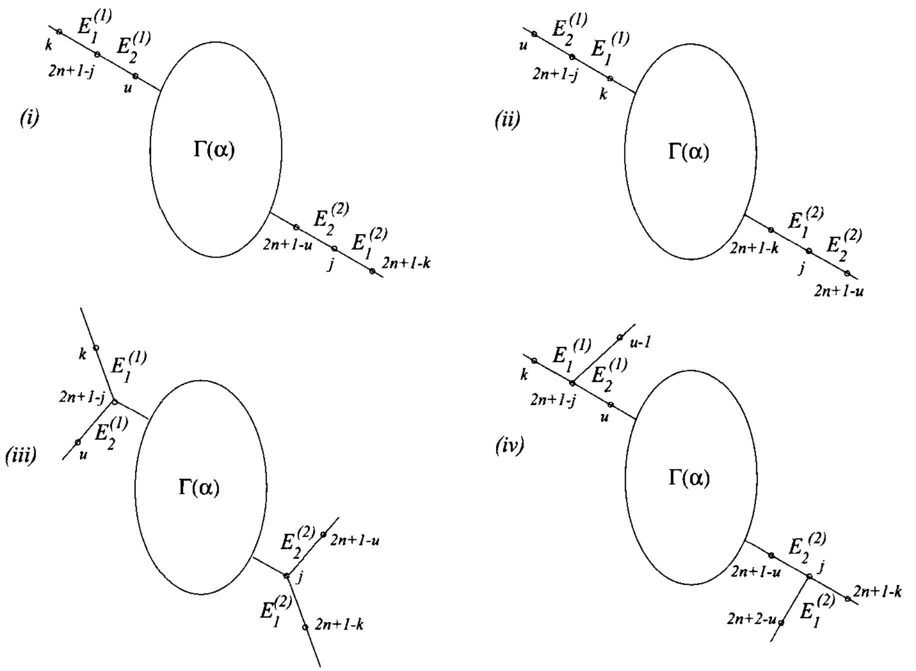

Figure 11.

If we have Figure 11 (i), then $u-1 = (u)\Pi_n = (2n+1-j)z_3\cdots z_n$ is in a component C of $\mathfrak{A}(\alpha) \setminus E_2^{(1)}$ which does not contain $\Gamma(\alpha)$ as shown in Figure 11 (iv). Now $j < u$ and so continually acting by Π_n shows that each of $u-1, u-2, \dots, j+1, j$ is in C. This is a contradiction as j is clearly not in C (see Figure 11 (i)). The exact same argument shows that Figure 11 (ii) does not occur either.

Now assume that we have Figure 11 (iii). Then $k-1=(k)\Pi_n=(u)z_3\cdots z_n$ is in a component C of $\mathfrak{A}(\alpha)\backslash E_2^{(1)}$ which does not contain $\Gamma(\alpha)$. Since $k<u$, acting by Π_n shows that $k-1,k-2,\ldots,j+1,j$ are all in C. This is a contradiction.

This proves (1), and (2), (3) follow as in Lemma 4.3 by acting by σ_1,σ_1^{-1}. ∎

Lemma 4.5 (1) concludes the proof of the $i=j$ subcase of case (i), (iii).

Now consider the situation $j<i<k$. Then we have $e_i.(e+j+e_k-e_{n-1}-e_n)=-2$ and so we need to show that the case $z_1=(i,n+2)(2n+1-i,n-1), z_2=(j,2n+1-k)(k,2n+1-j), 1\le i<j<k\le n-2$ cannot occur (noting that z_1,z_2 commute). This follows from the following result whose proof is similar to that of Lemma 4.4.

LEMMA 4.6: *If $\beta\in\mathcal{B}_n$ and $\alpha(Q_1,\ldots,Q_n)=(y_1,\ldots,y_n)$, then for $1\le k<n$ we cannot have $z_k=(i,2n+1-u)(2n+1-i,u), z_{k+1}=(j,2n+1-m)(2n+1-j,m)$ (or the other way around) where $j<i<m<n-1, 1\le u<n, m<u<n$.*

This does the case $j<i<k$, so next consider the case $i=k$. Here $e_k.(e_j+e_k-e_{n-1}-e_n)=-3$ and so we need to show that

$$z_1=(k,n+2)(2n+1-k,n-1),\quad z_2=(j,2n+1-k),(k,2n+1-j)$$

cannot occur. The techniques introduced so far suffice for this case as they do for the case $k<i$. This concludes case (i) (iii).

CASE (i), (iv): Here $v_1=\bar{a}(y_1)=e_i, v_2=\bar{a}(y_2)=e_j-e_{n-1}-e_n$ and we first consider the case $i<j$. The corresponding permutations are

$$z_1=(i,n+2)(2n+1-i,n-1),\quad z_2=(j,n-1)(2n+1-j,n+2).$$

Here $v_1.v_2=-1$ and dealing with this case amounts to showing that we cannot have $z_1,z_2,\ldots$ with z_i as above (one can easily show that the case $\bar{a}(y_2)=e_i$, $\bar{a}(y_1)=e_j-e_{n-1}-e_n$ is allowed). This follows from Lemma 4.5 (3).

Now assume that $i=j$. Then we have $v_1.v_2=-2$ and we must show that we cannot have

$$z_1=(i,n+2)(2n+1-i,n-1),\quad z_2=(i,n-1)(2n+1-i,n+2)$$

(or the other way around). But the permutations z_1,z_2 correspond to edges of $\mathfrak{A}(\alpha)$ which form a cycle: $E_1\cup E_2=\Gamma(\beta)$. However, this cycle does not contain the vertices $n,n+1$ and this contradicts Lemma 4.2 and so does this case.

Now assume that $j < i < n-1$. Then we have $v_1.v_2 = -3$ and we need to show that these cases never arise. The proof of this case is similar to that of Lemma 4.3 and is left to the reader.

If $i = n-1 > j$, then

$$z_1 = (n-1,n)(n+1,n+2), \quad z_2 = (j,n-1)(2n+1-j,n+2).$$

Since n is a vertex of E_1, it easily follows that both E_1 and E_2 are contained in $\Gamma(\beta)$, but that $E_1 \cup E_2 \neq \Gamma(\beta)$. One now considers the image of j; this is in a component C of $\mathfrak{A}(\beta) \setminus \{n-1\}$ and applying Π_n shows that C also contains $j-2,\dots,3,2,1,n+2$, which is a contradiction.

If $i = n$, then

$$z_1 = (n-1,n+1)(n,n+2), \quad z_2 = (j,n-1)(2n+1-j,n+2);$$

again E_1 and E_2 are contained in $\Gamma(\beta)$, but $E_1 \cup E_2 \neq \Gamma(\beta)$. Considering the image of $2n+1-j$ under Π_n we see that $2n+2-j$ belongs to a component C of $\mathfrak{A}(\beta) \setminus \{n+1\}$. Applying Π_n shows that C also contains $2n+3-j,\dots,2n,n-1$, a contradiction.

This concludes case (i), (iv) and so all cases with (i).

We now show how to reduce any other case to one of these. To do this we note that we can conjugate by the product $\mathcal{T}_n = T_1 \cdots T_n$ of the corresponding symplectic transvections. We will calculate this below. Thus to conclude the proof we will need to show that each of the vectors of types (ii)–(iv) is in the $\mathcal{T}_n$-orbit of some vector $e_i, 1 \leq i \leq n$.

Now to find $\mathcal{T}_n$ we first note that each T_i differs from the identity matrix in a single column:

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & \dots \\ -1 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 1 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \dots$$

$$T_{n-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 1 & 1 & 0 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}, \quad T_n = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}.$$

It then easily follows that their product is

$$\mathcal{T}_n = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 & 1 \\ -1 & 0 & 0 & 0 & \dots & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 0 & \dots & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 & \dots & 0 & -1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 1 \end{pmatrix}.$$

We thus need to show that each of the vectors of types (ii)–(iv) is the kth row of some power $\mathcal{T}_n^m$ for some k.

Now one checks that $\pm(e_i - e_j), i < j \leq n$ is a row of $\mathcal{T}_n^{-i}$, as is $\pm(e_i - e_{n-1} - e_n)$; that $\pm(e_i + e_{n-1-k} - e_{n-1} - e_n)$ is a row of $\mathcal{T}_n^k$ for $i < n-1-k < n-2$. This covers all cases and concludes the proof of Proposition 4.1 and so of the D_n case of Theorem 1.1. ∎

The cases I_3, I_4, F_4, E_6 are all calculated using [MA] as is Theorem 1.2.

5. Proof of Theorem 1.3

Let $G = \langle \underline{g} \rangle, \underline{g} = (g_1, \dots, g_n)$, be any finitely generated group. Let $F = \langle \underline{x} \rangle$, $\underline{x} = (x_1, \dots, x_n)$ be the free group of rank n and $\pi_G\colon F \to G, \pi(x_i) = g_i$, the quotient map. Let $\mathcal{B}_n(\underline{x}), \mathcal{B}_n(\underline{g})$ denote the orbit of $\underline{x}, \underline{g}$ (respectively) under the Hurwitz action. Then we have the permutation representation

$$\rho_G\colon \mathcal{B}_n \to \mathrm{Sym}(\mathcal{B}_n(\underline{g})).$$

Let $\mathcal{P}_G$ denote the image of ρ_G. Note that ρ_F is injective [Bi] and we will identify $\mathcal{P}_F$ with $\mathcal{B}_n$.

The epimorphism $\pi\colon F \to G$ induces an epimorphism

$$\pi_G^*\colon \mathcal{P}_F \to \mathcal{P}_G,$$

and we let $\mathcal{M}_G$ denote the kernel of π_G^*. Then for $\alpha \in \mathcal{M}_G$ we have $\pi_G(\alpha(x_i)x_i^{-1}) = 1_G$, so that $\alpha(x_i)x_i^{-1} \in K = \ker(\pi_G)$. Let

$$R_G = \{\alpha(x_i)x_i^{-1}\colon \alpha \in \mathcal{M}_G, 1 \leq i \leq n\},$$

and let $\bar{R}_G$ denote the normal closure of R_G in F and put $Q = F/\bar{R}_G$. Then we have $R_G \subseteq K$ and so we have an epimorphism $\pi\colon Q \to G$.

We next show that Q is an infinite group. Let $\zeta\colon F \to \mathbb{Z}$ be the epimorphism defined by $\zeta(x_i) = 1$ for $i \leq n$. Let $N = \ker(\zeta)$, so that N consists of all words in F with zero exponent. Note that for $\alpha \in \mathcal{B}_n$ the word $\alpha(x_i)x_i^{-1}$ has zero exponent and so is in N. Thus $R_G \subset N$ and so $\bar{R}_G \subset N$. Since N has infinite index in F it follows that $\bar{R}_G$ has infinite index in F. Thus Q is infinite.

Now $\pi\colon Q \to G$ induces an epimorphism $\chi_G\colon \mathcal{P}_Q \to \mathcal{P}_G$. To prove Theorem 1.3 it will suffice to show that χ_G is injective, which we now prove.

There is another epimorphism $\chi_Q = \pi_G^*\colon \mathcal{P}_F \to \mathcal{P}_Q$, induced by the epimorphism $F \to Q$, such that $\chi_G\chi_Q = \rho_G$. Let $\alpha_Q \in \ker\chi_G$. Since χ_Q is onto there is some $\alpha_F \in \mathcal{P}_F$ with $\chi_Q(\alpha_F) = \alpha_Q$. Then

$$\rho_G(\alpha_F) = \chi_G\chi_Q(\alpha_F) = \chi_G(\alpha_Q) = 1,$$

so that $\alpha_F \in \ker\rho_G$. It follows that $\pi_G(\alpha_F(x_i)x_i^{-1}) = 1$ for all $i \leq n$, so that $\alpha_F \in \mathcal{M}_G$ and thus $\alpha_F(x_i)x_i^{-1} \in R_G \subset \bar{R}_G$. It follows that $\alpha_F \in \ker\chi_Q$ so that $\alpha_G = \chi_Q(\alpha_F) = 1$, as required. ∎

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